A necessary and sufficient condition to play games in quantum mechanical settings

S K Özdemir, J Shimamura and N Imoto

SORST Research Team for Interacting Carrier Electronics,
CREST Research Team for Photonic Quantum Information,
Graduate School of Engineering Science, Osaka University, 1-3 Machikaneyama,
Toyonaka, Osaka 560-8531, Japan

E-mail: `ozdemir@qi.mp.es.osaka-u.ac.jp`

Abstract. Quantum game theory is a multidisciplinary field which combines quantum mechanics with game theory by introducing non-classical resources such as entanglement, quantum operations and quantum measurement. By transferring two-player-two strategy (2×2) dilemma containing classical games into quantum realm, dilemmas can be resolved in quantum pure strategies if entanglement is distributed between the players who use quantum operations. Moreover, players receive the highest sum of payoffs available in the game, which are otherwise impossible in classical pure strategies. Encouraged by the observation of rich dynamics of physical systems with many interacting parties and the power of entanglement in quantum versions of 2×2 games, it became generally accepted that quantum versions can be easily extended to N-player situations by simply allowing N-partite entangled states. In this article, however, we show that this is not generally true because the reproducibility of classical tasks in quantum domain imposes limitations on the type of entanglement and quantum operators. We propose a benchmark for the evaluation of quantum and classical versions of games, and derive the necessary and sufficient conditions for a physical realization. We give examples of entangled states that can and cannot be used, and the characteristics of quantum operators used as strategies.

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1. Introduction

Mathematical models and techniques of game theory have increasingly been used by computer and information scientists, i.e., distributed computing, cryptography, watermarking and information hiding tasks can be modelled as games [1, 2, 3, 4, 5, 6]. Therefore, new directions have been opened in the interpretation and use of game theoretical toolbox which has been traditionally limited to economical and evolutionary biology problems [7]. This is not a surprise because all have information as the common ingredient and the strong connection among them [8]: Game theory deals with situations where players make decisions which affect the outcomes and payoffs. All the involved processes can be modelled as information flow. Since physical systems,

which are governed by the laws of quantum mechanics, are used during information flow (generation, transmission, storage and manipulation), game theory becomes closely related to quantum mechanics, physics, computation and information sciences. Along this line of thinking, researchers introduced the quantum mechanical toolbox into game theory to see what new features will arise combining these two beautiful areas of science [9, 10, 11, 12, 13, 14, 15, 16, 17].

Quantum mechanics is introduced into game theory through the use of quantum bits (qubits) instead of classical bits, quantum operations and entanglement which is a quantum correlation with a highly complex structure and is considered to be the essential ingredient to exploit the potential power of quantum information processing. This effort, although has been criticized on the basis of using artificial models [18, 19], has produced significant results: (i) Dilemmas in some games can be resolved [20, 9, 16, 15, 21, 22], (ii) playing quantum games can be more efficient in terms of communication cost; less information needs to be exchanged in order to play the quantized versions of classical games [17, 8, 16], (iii) entanglement is not necessary for the emergence of Nash Equilibrium but for obtaining the highest possible sum of payoffs [16], and (iv) quantum advantage does not survive in the presence of noise above a critical level [23, 24]. In addition, market phenomena, bargaining, auction and finance have been described using quantum game theory [25]. The positive results are consequences of the fact that quantum mechanical toolbox allows players to have a larger set of strategies to choose from when compared to classical games.

In this paper, we focus on the extent of entangled states and quantum operators that can and cannot be used in multi-player games, and introduce a benchmark for the comparison of classical games and their quantized versions on a fair basis. Moreover, this study attempts to clarify a relatively unexplored area of interest in quantum game theory, that is the effect of different types of entangled states and their use in multi-player multi-strategy games in quantum settings. Our approach is based on the reproducibility of classical games in the physical schemes used for the implementation of their quantized versions.

Reproducibility requires that a chosen model of game should simulate both quantum and classical versions of the game to allow a comparative analysis of quantum and classical strategies, and to discuss what can or cannot be attained by introducing quantum mechanical toolbox. This is indeed what has been observed in quantum Turing machine which can simulate and reproduce the results of the classical Turing machine. Therefore, the reproducibility criterion must be taken into consideration whenever a comparison between classical and quantum versions of a task is needed. An important consequence of this criterion in game theory is the main contribution of this study: Derivation of the necessary and sufficient condition for entangled states and quantum operators that can be used in the quantized versions of classical games.

2. Multiplayer games

2.1. Definitions and model

In classical game theory, a strategic game is defined by $\Gamma = [N, (S_i)_{i\in N}, (\$_i)_{i\in N}]$ where N is the set of players, S_i is the set of pure strategies available to the i-th player, and $\$_i$ is his payoff function from the set of all possible pure strategy combinations $\mathcal{C} = \times_{j\in N} S_j$ into the set of real numbers $\mathbf{R}$. When the strategic game Γ is played with pure strategies, each player i choose only one of the strategies s_i from the set S_i. With each player having m pure strategies, $\mathcal{C}$ has m^N elements. Then for the k-th joint strategy $c_k \in \mathcal{C}$, payoffs of each player can be represented by an ordered vector $A_k = (a_k^1, a_k^2, ..., a_k^N)$ where $a_k^j = \$_j(c_k)$ is the payoff of the j-th player for the k-th joint strategy outcome. Players may choose to play with mixed strategies (randomizing among pure strategies) resulting in the expected payoff

$$f_i(\overline{q}_1, \cdot\cdot, \overline{q}_N) = \sum_{c_k\in\mathcal{C}} \left(\prod_{j\in N} q_j(s_j)\right) \$_i(c_k) = \sum_{k=1}^{m^N} \left(\prod_{j=1}^{N} q_j(s_j)\right) a_k^i \tag{1}$$

where $f_i(\overline{q}_1, \cdot\cdot, \overline{q}_N)$ is the payoff of the i-th player for the probability distributions $\overline{q}_t$ over the strategy set S_t of each player t, and $q_j(s_j)$ represents the probability that j-th player chooses the pure strategy s_j according to the distribution $\overline{q}_j$.

Most of the studies on quantum versions of classical games have been based on the model proposed by Eisert *et al.* [9]. In this model, the strategy set of the players consists of unitary operators which are applied locally on a shared entangled state. A measurement by a referee on the final state after the application of the operators maps the chosen strategies of the players to their payoffs. For example, the strategies "Cooperate" and "Defect" of players in classical Prisoner's Dilemma is represented by the unitary operators, $\hat{\sigma}_0$ and $i\hat{\sigma}_y$.

In this study, however, we consider a more general model than Eisert *et al.*'s model [9] for N-player-two-strategy games. In our model [26], (i) A referee prepares an N-qubit entangled state $|\Psi\rangle$ and distributes it among N players, one qubit for each player. (ii) Each player independently and locally applies an operator chosen from the entire set of special unitary operators for dimension two, SU(2), on his qubit. Assuming that i-th player applies $\hat{u}_i$, the joint strategy of all the players is represented by the tensor product of unitary operators as $\hat{x} = \hat{u}_1 \otimes \hat{u}_2 \otimes \cdots \otimes \hat{u}_N$, which generates the output state $\hat{x}|\Psi\rangle$ to be submitted to the referee. (iii) Upon receiving this final state, the referee makes a projective measurement $\{\hat{\mathcal{P}}_j\}_{j=1}^{2^N}$ which outputs j with probability $\mathrm{Tr}[\hat{\mathcal{P}}_j \hat{x}|\Psi\rangle\langle\Psi|\hat{x}^\dagger]$, and assigns payoffs chosen from the payoff matrix depending on the measurement outcome j. Therefore, the expected payoff of the i-th player is described by

$$f_i(\hat{x}) = \mathrm{Tr}\left[\left(\Sigma_j a_j^i \hat{\mathcal{P}}_j\right) \hat{x}|\Psi\rangle\langle\Psi|\hat{x}^\dagger\right] \tag{2}$$

where $\hat{\mathcal{P}}_j$ is the projector and a_j^i is the i-th player's payoff when the measurement outcome is j. This model can be implemented in a physical scheme with the current level of quantum technology.

2.2. Classification of N-player two-strategy games:

In general, one can prepare a large number of generic games by arbitrarily choosing the entries of game payoff matrix. However, not all of those generic games are interesting enough to be the subject of game theory. Classical game theory mainly focuses on specific dilemma containing 2×2 games such as Prisoner's dilemma (PD), Stag-Hunt (SH), Chicken Game (CG), Dead-Lock (DL), Battle of Sexes (BoS) Samaritan's dilemma (SD), Boxed Pigs (BP), Modeller's dilemma (MD), Ranked Coordination (RC), Alphonse & Gaston Coordination Game (AG), Hawk-Dove (HD), Battle of Bismarck (BB), Matching Pennies (MP) [7]. Multi-player extensions of these 2×2 games and some originally multi-player games, such as minority and coordination games which have direct consequences where populations are forced to coordinate their actions, are also the subject of game theory. In this study, we consider only those interesting games instead of studying all generic games that can be formed.

In an N-player game, every player plays one of his strategies against all other $N-1$ players simultaneously. The payoff matrix of an N-player two-strategy game is characterized by 2^N possible outcomes and a total of $N2^N$ parameters. Payoffs of each player for the k-th possible outcome can be represented by an ordered vector $A_k = (a_k^1, a_k^2, ..., a_k^N)$. Based on the payoffs for all possible outcomes, we group the games into two: *Group I* contains the games where all the outcomes have different payoff vectors, that is $A_j \neq A_k$ for $\forall k \neq j$, whereas *Group II* contains the games where payoff vectors for some outcomes are the same, $A_j = A_k$, implying $(a_j^1, a_j^2, ..., a_j^N) = (a_k^1, a_k^2, ..., a_k^N)$, for $\exists k \neq j$.

When a two-player two-strategy game is extended to N-player game ($N > 2$), the new payoff matrix is formed by summing the payoffs that each player would have received in simultaneously playing the two-player game with $N-1$ players. Hence, in their N-player extensions, the games PD, SD, BP, MD, DL, and RC fall into *Group I* while BoS, BB, MP, and AG games in *Group II.* For $N = 3$, BoS becomes a member of *Group I.* CG, SH and HD belong to either the first or second group according to whether N is even or odd. For even N, SH belongs to *Group I* and CG and HD belong to *Group II*, and vice verse. Minority, majority and coordination Games are in *Group II.*

3. Reproducibility criterion to play games in quantum mechanical settings

We consider the reproducibility of a multi-player two-strategy classical game in the quantization model explained above. First, reproducibility problem in pure strategies will be discussed in details, and later the conditions for mixed strategies will be given. We require that a classical game be reproduced when each player's strategy set is restricted to two unitary operators, $\{\hat{u}_i^1, \hat{u}_i^2\}$, corresponding to the two pure strategies in the classical game. Then the joint pure strategy of the players is represented by $\hat{x}_k = \hat{u}_1^{l_1} \otimes \hat{u}_2^{l_2} \otimes \cdots \otimes \hat{u}_N^{l_N}$ with $l_i = \{1, 2\}$ and $k = \{1, 2, \ldots, 2^N\}$. Thus the output state

becomes $|\,\Phi_k\rangle = \hat{x}_k|\,\Psi\rangle$. For the strategy combination $\hat{x}_k$, expected payoff for the i-th player becomes as in Eq. (2) with $\hat{x}$ replaced by $\hat{x}_k$. Then A_k defined in the previous section is the ordered payoff vector of all players for the k-th possible outcome.

Reproducibility problem can be stated in two cases: In *CASE I*, the referee should be able to identify the strategy played by each player deterministically regardless of the structure of the payoff matrix, whereas in *CASE II* the referee should be able to reproduce the expected payoff given in eq. (1) in the quantum version, too, [26]. While in *CASE I* the referee needs to identify all possible outcomes, in *CASE II* he just needs to distinguish between the sets of outcomes with the same payoff. *CASE II* is equivalent to *CASE I* for *Group I* games where all outcomes of the game have different payoff vectors. We call the situations described in *CASE I* and its equivalence in *CASE II* as the "strong criterion," and the rest of the situations as the "weak criterion" of reproducibility.

3.1. The strong criterion of reproducibility (SCR)

This criterion requires that referee discriminate all the possible output states $|\,\Phi_k\rangle$ deterministically in order to assign payoffs uniquely in the pure strategies. That is, the projector $\{\hat{\mathcal{P}}_j\}_{j=1}^{2^N}$ has to satisfy $\mathrm{Tr}[\hat{\mathcal{P}}_j|\,\Phi_k\rangle\langle\Phi_k\,|] = \delta_{jk}$, which is possible if and only if

$$\langle\Phi_\alpha\,|\,\Phi_\beta\rangle = \delta_{\alpha\beta}\ \ \forall\alpha,\beta. \tag{3}$$

Thus, SCR transforms the reproducibility problem into quantum state discrimination where we know that two quantum states can be deterministically discriminated iff they are orthogonal. Under SCR, we see that $f_i(\overline{q}_1,\cdot\cdot,\overline{q}_N) = f_i(\hat{x}_k) = a_k^i$ because there is no randomization over the strategy sets (each player choose one and only one strategy deterministically) and the only outcome is $\hat{x}_k$ with probability one. Therefore, Eq. (3) becomes the *necessary condition* for the strong reproducibility criterion (SCR).

Among the multi-partite ($N\geq 3$) entangled states we focused on GHZ-like states of the form $|\,\mathrm{GHZ}\rangle_N = (|\,00\ldots0\rangle + i|\,11\ldots1\rangle)/\sqrt{2}$ and symmetric Dicke states represented as $|\,N-m,m\rangle/\sqrt{{}_NC_m}$ with $(N-m)$ zeros and m ones (${}_NC_m$ denoting the binomial coefficient). Imposing SCR we observed [26] the following.

(a) For Dicke states with unequal number of zeros and ones (N-party W-state, defined as $|W_N\rangle = |N-1,1\rangle/\sqrt{N}$ is a member of this class),

- (*a1*) $\hat{u}_k^{1\dagger}\hat{u}_k^2 = \hat{\sigma_x}\hat{R}_z(2\phi_k)$ for any two output states differing only in k-th player's strategy where the rotation operator $\hat{R}_z(\gamma)$ is defined as $\hat{R}_z(\gamma) = e^{-i\gamma\hat{\sigma}_z/2}$, and
- (*a2*) $\phi_j - \phi_k = n\pi + \pi/2$ for any two output states different only in the strategies of j-th and k-th players.

Then for any three players j,k,m participating the game, we obtain the set of equations $\chi_{jkm} = \{\phi_j - \phi_k = n\pi + \pi/2,\ \phi_m - \phi_j = n'\pi + \pi/2,\ \phi_k - \phi_m = n''\pi + \pi/2\}$ where n, n' and n'' are integer. The sum of the three equations in χ_{jkm} results in $3\pi/2 + m'\pi = 0$ which is satisfied for $m' = -3/2$; however this contradicts the fact $m' = n + n' + n''$ is an integer.

(b) For Dicke states $|N/2, N/2\rangle$ with even $N \geq 6$,

(*b1*) $\hat{u}_k^{1\dagger}\hat{u}_k^2 = \cos\theta_k\hat{\sigma}_z + \sin\theta_k\hat{\sigma}_x\hat{R}_z(2\phi_k)$ with real θ_k and ϕ_k is a two-parameter SU(2) operator obtained from the mutual orthogonality of two output states which differ in the strategies of one player,

(*b2*) $\cos\theta_k\cos\theta_j = (N/2)\cos(\theta_k - \theta_j)\sin\theta_k\sin\theta_j$ from the inner product of two states which differ only in the strategies of two-players, and

(*b3*) from the output states which differ in the strategies of four players i, j, k, l,

$$\frac{24}{N(N-2)}\cos\theta_i\cos\theta_j\cos\theta_k\cos\theta_l$$
$$= [\cos\beta_1 + \cos\beta_2 + \cos\beta_3]\sin\theta_i\sin\theta_j\sin\theta_k\sin\theta_l \tag{4}$$

where $\beta_1 = \phi_i + \phi_j - \phi_k - \phi_l$, $\beta_2 = \phi_i - \phi_j + \phi_k - \phi_l$ and $\beta_3 = \phi_i - \phi_j - \phi_k + \phi_l$.

Then we obtain $\theta_i \neq n\pi$ and $\theta_i \neq \pi/2 + n\pi$ for $\forall i$ using (*b1,b2*) and (*b1,b2,b3*), respectively. Next, we write (*b2*) for the pair of players (i, j) and (k, l) and multiply these two equations. Doing the same for different pairs of players (i, k) and (j, l), and comparing the final expressions with Eq. 4, we find $\theta_i = \pi/2 + n\pi$ for $\forall i$ which contradicts the above result obtained from (*b1,b2,b3*).

If the mutual orthogonality relations lead to *contradictions* outlined in (a) and (b), the corresponding entangled state cannot be used in quantum versions of classical games under SCR. Among the class of entangled states studied we have found: (i) bell states and any two-qubit pure state satisfy SCR if the unitary operators for the players are chosen as $\{\hat{\sigma}_0, \hat{\sigma}_x\}$ and $\{\hat{\sigma}_0, i\hat{\sigma}_y\}$. (ii) $|\,\mathrm{GHZ}\rangle_N$ satisfies SCR if the unitary operators of the players are chosen as $\{\hat{\sigma}_0, i\hat{\sigma}_y\}$. Entangled states that can be obtained from $|\,\mathrm{GHZ}\rangle_N$ state by local unitary transformations also satisfy SCR. (iii) $|W_N\rangle$ does not satisfy SCR, therefore cannot be used in this model of quantum games. (d) Among the Dicke states, only the states $|\,1, 1\rangle$ and $|\,2, 2\rangle$ satisfy the SCR. These results are valid for all the games in *Group I* and the situations where *CASE I* is desired.

3.1.1. Quantum operators and SCR Assume that there are two unitary operators corresponding to the classical pure strategies for the entangled state, $|\,\Psi\rangle$. Imposing SCR on the situation where two outcomes, $|\,\Phi_0\rangle = \hat{u}_1^1 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1|\,\Psi\rangle$ and $|\,\Phi_1\rangle = \hat{u}_1^2 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1|\,\Psi\rangle$, differ only in the operator of the first player, we find

$$\langle\Psi\,|(\hat{u}_1^1)^\dagger\hat{u}_1^2 \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi\rangle = 0. \tag{5}$$

Since $(\hat{u}_1^1)^\dagger\hat{u}_1^2$ is a normal operator, it can be diagonalized by a unitary operator $\hat{z}_1$. Furthermore, since $(\hat{u}_1^1)^\dagger\hat{u}_1^2$ is a SU(2) operator, the eigenvalues are given by $e^{i\phi_1}$ and $e^{-i\phi_1}$. Then Eq. (5) can be transformed into

$$\langle\Psi\,|\,\hat{z}_1^\dagger\,(\hat{z}_1(\hat{u}_1^1)^\dagger\hat{u}_1^2\hat{z}_1^\dagger)\,\hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I}\,|\,\Psi\rangle$$
$$= \langle\Psi'\,|R_z(-2\phi_1) \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi'\rangle$$
$$= \cos\phi_1 + i\left(2\Sigma_{i_j\in\{0,1\}}|c_{0\,i_2\ldots i_N}|^2 - 1\right)\sin\phi_1 = 0 \tag{6}$$

where $|\,\Psi'\rangle = \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi\rangle$ is written on computational basis as $|\,\Psi'\rangle = \Sigma_{i_j\in\{0,1\}}c_{i_1\,i_2\ldots i_N}|\,i_1\rangle|\,i_2\rangle\cdots|\,i_N\rangle$.

In order for the above equality to hold, $\cos\phi_1 = 0$ and $2\Sigma_{i_j \in \{0,1\}} |c_{0\,i_2 \ldots i_N}|^2 - 1 = 0$ must be satisfied. The equation $\cos\phi_1 = 0$ implies that the diagonalized form $\hat{D}_1 = \hat{z}_1 (\hat{u}_1^1)^\dagger \hat{u}_1^2 \hat{z}_1^\dagger = i\hat{\sigma}_z$. This argument holds for all players, therefore we write $\hat{z}_k (\hat{u}_k^1)^\dagger \hat{u}_k^2 \hat{z}_k^\dagger = \hat{D}_k = i\hat{\sigma}_z$ for $k = 1, \cdots, N$. For example, in the case of the Dicke state $|\,2,2\rangle$, which satisfies SCR with the unitary operators $\hat{u}_{k=1,2,3,4}^1 = \hat{I}$, $\hat{u}_{k=1,2,3}^2 = i(\sqrt{2}\hat{\sigma}_z + \hat{\sigma}_x)/\sqrt{3}$ and $\hat{u}_4^2 = i\hat{\sigma}_y$, it is easy to verify that eigenvalues $\mp i$ of ${\hat{u}_k}^{1\dagger} \hat{u}_k^2$ are already in the diagonalized form. For GHZ state, the operators are $\hat{u}_k^1 = \hat{I}$ and $\hat{u}_k^2 = i\hat{\sigma}_y$ which can be written in the form $\hat{D}_1$.

Next we consider the following scenario: Each player has two operators satisfying the above properties. Instead of choosing either of these operators, they prefer to use a linear combination of their operator set. Let this operator be $\hat{w}_k = \hat{u}_k^1 \cos\theta_k + \hat{u}_k^2 \sin\theta_k$ for the k-th player. Then, we ask (i) Does the property of the operators $\hat{u}_k^1$ and $\hat{u}_k^2$ derived from the SCR impose any condition on the operator $\hat{w}_k$?, and (ii) What does the outcome of the game played in the quantum version with the operator $\hat{w}_k$ imply? Since $\hat{z}_k (\hat{u}_k^1)^\dagger \hat{u}_k^2 \hat{z}_k^\dagger$ is in the diagonalized form we can write

$$\hat{w}_k^\dagger \hat{w}_k = \hat{I} + \cos\theta_k \sin\theta_k (\hat{z}_k^\dagger \hat{D}_k \hat{z}_k + \hat{z}_k^\dagger \hat{D}_k^\dagger \hat{z}_k) = \hat{I}, \tag{7}$$

where we have used ${\hat{u}_k}^{1\dagger} \hat{u}_k^2 = \hat{z}_k^\dagger \hat{D}_k \hat{z}_k$, and $\hat{D}_k^\dagger = -\hat{D}_k$ since $\hat{D}_k$ is anti-hermitian. This equation implies that SCR requires $\hat{w}_k$ be a unitary operator.

When the players use the operators $\hat{w}_k = \hat{u}_k^1 \cos\theta_k + \hat{u}_k^2 \sin\theta_k$, the joint strategy $\hat{x}$ becomes $\hat{x} = \hat{w}_1 \otimes \hat{w}_2 \otimes \ldots \otimes \hat{w}_N = \bigotimes_{j \in N} \hat{w}_j$. Substituting $\hat{x}$ into Eq. (2), we obtain

$$f_i(\hat{x}) = \sum_{\mu=1}^{2^N} \left(\prod_{\ell=1}^{\mu-1} \sin^2\theta_\ell \right) \left(\prod_{j=\mu}^{N} \cos^2\theta_j \right) a_\mu^i. \tag{8}$$

Note that Eq. (8) has the same form of Eq. (1) implying that payoffs of the classical mixed strategies are reproduced in the quantum version for $\hat{w}_k$. Therefore, we conclude that Eq. (3) is the *necessary and sufficient condition* for the reproducibility of a classical game in the quantum version according to SCR. This is because when players apply one of their pure strategies $\hat{u}_k^1$ or $\hat{u}_k^2$ with unit probability, results of classical pure strategy; when they apply a linear combination of their pure strategies results of classical mixed strategy are reproduced in the quantum setting. Another way of reproducing the results of classical mixed strategies is that players apply their pure strategies $\hat{u}_k^1$ and $\hat{u}_k^2$ according to a probability distribution as is the case in classical mixed strategies. Note that this is different than applying a linear combination of their pure strategies $\hat{u}_k^1$ and $\hat{u}_k^2$.

3.1.2. Entangled states and SCR After stating the properties of operators which satisfy SCR, we proceed to investigate the properties of the class of entangled states which satisfy it. Suppose that an N-qubit state $|\,\Psi\rangle$ and two unitary operators $\{\hat{u}_k^1, \hat{u}_k^2\}$ satisfy SCR. Then for two possible outcomes $|\,\Phi_0\rangle = \hat{u}_1^1 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1 |\,\Psi\rangle$ and $|\,\Phi_1\rangle = \hat{u}_1^2 \otimes \hat{u}_2^1 \otimes \cdots \otimes \hat{u}_N^1 |\,\Psi\rangle$, Eq. (3) requires

$$\langle \Psi\,|\hat{z}_1^\dagger\, \hat{z}_1 {\hat{u}_k}^{1\dagger} \hat{u}_k^2 \hat{z}_1^\dagger\, \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi\rangle = \langle \Psi'\,|\hat{D}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I} |\,\Psi'\rangle = 0, \tag{9}$$

where $\hat{z}_1$ is a unitary operator diagonalizing $\hat{u}_1^{1\dagger}\hat{u}_1^2$ and $|\,\Psi'\rangle = \hat{z}_1 \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi\rangle$. This implies that if the N-qubit state $|\,\Psi\rangle$ and the operators $\{\hat{u}_k^1, \hat{u}_k^2\}$ satisfy Eq. (3), then the state $|\,\Psi'\rangle = \hat{z}_1 \otimes \hat{z}_2 \otimes \cdots \otimes \hat{z}_N|\,\Psi\rangle$ and the unitary operators $\{\hat{D}, \hat{I}\}$ should satisfy, too. Since the global phase is irrelevant, Eq. (9) can be further reduced to $\langle\Psi'\,|\hat{\sigma}_z \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi'\rangle = 0$. Thus, we end up with $2^N - 1$ equalities to be satisfied:

$$\begin{aligned}
&\langle\Psi'\,|\hat{\sigma}_z \otimes \hat{I} \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi'\rangle = 0,\\
&\langle\Psi'\,|\hat{I} \otimes \hat{\sigma}_z \otimes \hat{I} \otimes \cdots \otimes \hat{I}|\,\Psi'\rangle = 0,\\
&\qquad\quad\vdots\\
&\langle\Psi'\,|\hat{\sigma}_z \otimes \hat{\sigma}_z \otimes \cdots \otimes \hat{\sigma}_z|\,\Psi'\rangle = 0.
\end{aligned} \tag{10}$$

Defining $|\,\Psi'\rangle = \Sigma_{i_j\in\{0,1\}} c_{i_1 i_2 \ldots i_N}|\,i_1\rangle|\,i_2\rangle\cdots|\,i_N\rangle$, we write Eq. (9) in the matrix form as

$$\begin{bmatrix} 1 & 1 & \ldots & -1 & -1 \\ & & \cdots & & \\ & & \vdots & & \\ 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \begin{bmatrix} |c_{00\ldots0}|^2 \\ |c_{00\ldots1}|^2 \\ \vdots \\ |c_{11\ldots1}|^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{11}$$

where the last row is the normalization condition. The row vector corresponds to the diagonal elements of $\hat{\sigma}_z^{\{0,1\}} \otimes \cdots \otimes \hat{\sigma}_z^{\{0,1\}}$ where $\hat{\sigma}_z^0$ is defined as $\hat{I}$. Consider the operators $\hat{x}, \hat{y} \in (\hat{\sigma}_z^{\{0,1\}})^{\otimes N}$ where $\hat{x}\hat{y} \in (\hat{\sigma}_z^{\{0,1\}})^{\otimes N}$. Since $\mathrm{Tr}[\hat{\sigma}_z] = 0$, for $\hat{x} \neq \hat{y}$, we have $\mathrm{Tr}[\hat{x}\,\hat{y}] = \mathrm{Tr}[\hat{x}]\,\mathrm{Tr}[\hat{y}] = 0$. Thus any two row vectors are orthogonal to each other, thus the matrix in Eq. (11) has an inverse, and $|c_{i_1 i_2 \ldots i_N}|^2$ are uniquely determined as $1/N$. This implies that if a state satisfies SCR, then it should be transformed by local unitary operators into the state which contains all possible terms with the same magnitude but different relative phases:

$$|\,\Psi'\rangle = \frac{1}{\sqrt{N}} \sum_{i_j\in\{0,1\}} e^{i\phi_{i_1 i_2 \ldots i_N}}|\,i_1\rangle|\,i_2\rangle\cdots|\,i_N\rangle. \tag{12}$$

One can show that product state and GHZ state, which satisfy SCR, can be transformed into the form of Eq. (12), respectively, by Hadamard operator, $\hat{H} = (\hat{\sigma_x} + \hat{\sigma_z})/\sqrt{2}$, and by $(e^{i\frac{\pi}{4}}\hat{I} + e^{-i\frac{\pi}{4}}\hat{\sigma}_z + \hat{\sigma}_y)/\sqrt{2}$ for one player and $\hat{H}$ for the others.

3.2. The weak criterion of reproducibility (WCR)

This weak version of the reproducibility criterion requires that referee deterministically discriminate all the possible sets formed by the output states with the same payoff vectors in order to assign payoffs uniquely in the pure strategies. When $A_j = A_k$, output states $|\,\Phi_j\rangle$ and $|\,\Phi_k\rangle$ should be grouped into the same set. If all possible output states are grouped into sets $S_j = \{|\Phi_{1j}\rangle, |\Phi_{2j}\rangle, \ldots |\Phi_{nj}\rangle\}$ and $S_k = \{|\Phi_{1k}\rangle, |\Phi_{2k}\rangle, \ldots |\Phi_{n'k}\rangle\}$ then the referee should deterministically discriminate between these sets which is possible iff the state space spanned by the elements of each set are orthogonal. Hence for every element of S_j and S_k, we have $\langle\Phi_{nj}|\Phi_{n'k}\rangle = 0, \forall j \neq k$, that is all the elements of S_j and S_k must be orthogonal to each other, too. Thus WCR transforms the reproducibility problem into set discrimination problem. We named it as WCR because the condition of

sets $S_1, .., S_k$ being mutually orthogonal to each other is a much looser condition than the condition of all states in $S = \bigoplus_i^k S_i$ being mutually orthogonal to each other. The sets $S_1, .., S_k$ may be mutually orthogonal even if the states in S are linearly dependent. If we relax the criterion of deterministic discrimination and allow inconclusive results then one can use unambiguous state and set discrimination. However, we are not concerned with this situation because we require that classical game is reproduced in the quantum settings deterministically. It is clear that the games in *Group II* should be discussed with WCR. A natural question is whether the results listed in (a)-(d) are valid for *Group II* games or not. The answer to this question will be given below.

3.2.1. Entangled states and WCR In this section we check whether the results obtained under SCR is valid or not for *Group II* games with WCR. We start by asking the question "*Is there a partition of all possible outcomes (output states) into sets such that mutual orthogonality of these sets does not lead to the contradictions discussed for SCR?*" The following observations from the analysis of SCR for a given entangled state makes our task easier:

(*O1*) For $|N-m,m\rangle$ with $N \neq m$, if the mutual orthogonality condition of the sets leads to the operator form as in (*a1*) for all players, then there will be *contradiction* if we obtain the set χ_{jkm} for any three-player-combination (j,k,m). Presence of at least one such set is enough to conclude that there is *contradiction*. On the other hand, to prove that there is no contradiction, one has to show that at least one of the equations in χ_{jkm} is missing for all three-player-combinations.

(*O2*) For $|N-m,m\rangle$ with $N \neq m$, if the mutual orthogonality condition of the sets leads to the operator form as in (*a1*) for one and only one player, then there will be *no contradiction* because there will be at least one missing equation in χ_{jkm} for all possible three-player-combinations (j,k,m). Note that such a situation occurs iff 2^N possible outputs are divided into two sets with equal number of elements. Then the only equations we will obtain are $\phi_1 - \phi_j = n\pi + \pi/2$ for all $j = 2, \cdots, N$.

(*O3*) For $|N/2, N/2\rangle$ with $N \geq 6$, a necessary condition for contradiction is to have equations of the form (*b2*) for at least four different pairing of players, such as $\{(i,j),(k,l)\}$ and $\{(i,k),(j,l)\}$. If the mutual orthogonality condition of the sets leads to the operator form as in (*b1*) for one and only one player, say first player, then from (*b2*) we will obtain only $\cos\theta_1 \cos\theta_j \exp(\varphi_j) = (N/2)\cos(\theta_1 - \theta_j)\sin\theta_1 \sin\theta_j$ for all $j = 2, \cdots, N$ where $\exp(\mp\varphi_j)$ denotes the phase of the diagonal elements of the matrix $u_j^{1\dagger} u_j^2$. This extra phase parameter and the absence of similar relations between players other than the first allow us to freely set the operator parameters for all players. Therefore, *no contradiction* occurs.

(*O4*) For $|N/2, N/2\rangle$ with $N \geq 4$, if the mutual orthogonality of states leads to the relations in (*b1*) and (*b2*) then contradiction will not occur iff the outcomes differing in the strategies of four players are in the same set.

(*O5*) For all $|N-m,m\rangle$ except $|1,1\rangle$ and $|2,2\rangle$, if the number of elements in any of the sets in a *Group II* game is an odd number, then there will always be

contradiction. If one of the output states in any set is left alone then this state will satisfy the mutual orthogonality condition with the elements of the other sets which will lead to the relations mentioned above, and hence to *contradiction.*

(*O6*) If there is a set with only two elements which are the outcomes when all the players choose the same strategy, there will be *contradiction.*

Our analysis revealed that multiparty extensions of 2×2 games have payoff structures such that partitioning results in one or more sets with only one element. The number of sets with one element depends on the payoff matrix and the number of players participating the game. Therefore, based on the above observations, especially (*O5*), we can immediately conclude that for multiparty extensions of 2×2 games classified into *Group II*, there will always be a contradiction for the states $|W_N\rangle$ and $|\,N-m, m\rangle/\sqrt{{}_N C_m}$ except for $|\,1,1\rangle/\sqrt{2}$ and $|\,2,2\rangle/\sqrt{6}$. Hence, the results obtained for SCR are valid for WCR as well.

3.2.2. Multiplayer games according to WCR In the previous subsection, we showed that the results of SCR are valid in case of WCR for multiparty extensions of two-player two-strategy games. Here, we consider the class of games which are originally designed as multiplayer games:

For the Minority game, the payoff structure is such that there is no set with odd number of elements therefore we cannot exploit (*O5*). However, we have (*O1*) which is valid for $|W_N\rangle$ and Dicke states $|\,N-m, m\rangle$ with $N \neq m$. For the Dicke states with $N = m$, the situation mentioned in (*O4*) occurs only for the state $|\,2,2\rangle$ because pairs of output states leading to relations as in Eq. 4 are in the same sets. Hence, for this state there will be *no contradiction.* On the other hand, when $N \geq 6$ the type of *contradictions* described in (*b1*)-(*b3*) are seen. Hence, the results obtained for the case of SCR are valid for Minority game. Consider $N = 4$ for which the payoff structure imposes the partitions $S_1 = \{\phi_{1,4,6,7,10,11,13,16}\}$, $S_2 = \{\phi_{2,15}\}$, $S_3 = \{\phi_{3,14}\}$, $S_4 = \{\phi_{5,12}\}$ and $S_5 = \{\phi_{8,9}\}$. The outcomes differing with the strategies of four players are in the same sets. Therefore, for the Dicke state with $N = m = 2$ there will be no contradiction and this state can be used. For $|W_4\rangle$, mutual orthogonality of set-pairs $(S_1, S_{3,4,5})$ requires $\langle\phi_1|\phi_{3,5,9}\rangle = 0$ which gives $\hat{u}_k^{1\dagger}\hat{u}_k^2 = \hat{\sigma}_x\hat{R}_z(2\phi_k)$ for $\forall k$. Substituting in the orthogonality relations from $(S_5, S_{2,3,4})$ we obtain $\langle\phi_{2,3,5}|\phi_8\rangle = 0$ which gives χ_{234} implying a *contradiction.*

In a *coordination game*, the players receive the payoffs $\lambda_0 > 0$ ($\lambda_1 > 0$), when all choose the first (second) strategy; otherwise, they receive zero. If $\lambda_0 \neq \lambda_1$, players make their choices for the strategy with the higher payoff. A game-theoretic situation occurs only when $\lambda_0 = \lambda_1$, because players cannot coordinate their moves without communication. The payoff structure and outcomes of such a game can be grouped into two sets; the first one will have two elements, where all players choose either the first or second strategy, $S_1 = \{\phi_1, \phi_{2^N}\}$, and the second one, S_2 will have the rest of the outcomes. In such a partition all the contradictions mentioned above will appear except for the state $|2,2\rangle$. If $\lambda_0 \neq \lambda_1$ then we will have three sets two of which will be with one

element, and hence the observation (*O5*) will be valid. Therefore, we conclude for this game and any other game with such a payoff structure, all the results of SCR are valid.

For a *majority game*, all the players receive λ_0 or λ_1 depending on whether the majority is achieved in the first or second strategy, respectively. In case of even-split all get zero. Outcomes are grouped into three and four sets for odd and even N, respectively. For both cases all the results obtained for SCR is valid. Here we give the examples for $N = 3$ and $N = 4$. When $N = 3$, outcomes are grouped as $S_1 = \{\phi_{1,2,3,5}\}$ and $S_2 = \{\phi_{4,6,7,8}\}$. From $\langle\phi_{2,3}|\phi_4\rangle$ and $\langle\phi_2|\phi_6\rangle$ we obtain $\hat{u}_k^{1\dagger}\hat{u}_k^2 = \hat{\sigma}_x\hat{R}_z(2\phi_k)$ for $\forall k$. Then $\langle\phi_1|\phi_{4,6}\rangle$ and $\langle\phi_2|\phi_8\rangle$ results in χ_{123}. This is exactly the situation in (*O1*). For $N = 4$, the outcomes are divided into three sets as $S_1 = \{\phi_{1,2,3,5,9}\}$, $S_2 = \{\phi_{8,12,14,15,16}\}$ and $S_3 = \{\phi_{4,6,7,10,11,13}\}$ where S_3 has the outcomes for the even-split of choices. For $|W_4\rangle$ and $|\,N-m, m\rangle$ with $N \neq m$, we have $\hat{u}_k^{1\dagger}\hat{u}_k^2 = \hat{\sigma}_x\hat{R}_z(2\phi_k)$ for $\forall k$ from $\langle\phi_2|\phi_{4,6,10}\rangle = 0$ and $\langle\phi_3|\phi_4\rangle = 0$ due to the orthogonality of (S_1, S_3). Moreover, we have $\langle\phi_1|\phi_{4,6,7}\rangle = 0$ which results in the set χ_{234}. This is also exactly the situation in (*O1*). Similar contradiction can be obtained from the orthogonality of $(S2, S3)$, too. On the other hand when we use $|2,2\rangle$, one can show that no contradictions occur and the strategies can be chosen as $\hat{u}_k^1 = \hat{\sigma}_0$, $\hat{u}_1^2 = \hat{\sigma}_x$ and $\hat{u}_2^2 = \hat{u}_3^2 = \hat{u}_4^2 = (\sqrt{2}\hat{\sigma}_z + \hat{\sigma}_y)/\sqrt{3}$.

In a *zero-sum game* where there is competitive advantage λ, if all players choose the same strategy, there is no winner and loser so all receive zero. Otherwise, each of the m players choosing the first strategy gets λ/m, and the rest of the players loses $\lambda/(N-m)$. The outcomes are grouped into $2^N - 1$ sets where one set has two elements obtained when all players choose the same strategy and the rest with one element. In this case, (*O6*) is valid and hence there will be *contradiction*. In the multi-player extension of MP game, the outcomes when all players choose the same strategy are always grouped into one set of two elements, and the rest of the outcomes are grouped in sets of even-number of elements (the number of sets depends on N). Thus, (*O6*) is observed, and hence the same results are valid.

A *symmetric game* with a strict ordering of the payoffs is a *Group I*; otherwise a *Group II* game. We analyzed such games up to $N = 6$, and found that all the results concerning the entangled states and operators are valid except for a few exceptional cases which we could not relate to any game-theoretic situation when $N = 3$ and $N = 6$. For $N = 3$, we have eight outcomes with the payoff vectors as (a, a, a), (b, b, d), (b, d, b), (c, e, e), (d, b, b), (e, c, e), (e, e, c) and (f, f, f). With proper choices of the parameters, one can obtain multiplayer extensions of the symmetric games, PD, MD, RC, CG, SH and AG. For other possible generic games, we search for the values of the payoff entries for which there will be no contradiction according to discussions above. For the entangled states $|W_3\rangle$ and $|\,3-m, m\rangle/\sqrt{{}_3C_m}$, we know from (*O5*) that all sets must have even number of elements. We identify five possible partitions (2 two-set partitions and 1 one-, three- and four- set partitions): (1) One- and four-set partitions require all outcomes be the same, $a = b = c = d = e = f$, that is all players receive the same payoff no matter which strategy they choose. This is not a game. (2) Three-set partitions result in three different cases: (i) $a = b = d$ and $c = e = f$ which is the majority game

discussed above, (ii) $a = b = c = d = e = f$ as in (1), and (iii) $a = c = e = \lambda_0$ and $b = d = f = \lambda_1$ where payoffs of the players are equal regardless of their choice. Players receive λ_0 when two-players choose the second strategy and one chooses the first or when they all choose the first strategy; otherwise they receive λ_1. Such a situation does not correspond to a game-theoretic one. (3) Two-set partitions, in addition to those listed in (2), result in $a = f$ and $b = d = c = e$ which corresponds to coordination game discussed above. In the case of the Dicke state For $|\, N/2, N/2\rangle$ with $N = 6$, no contradiction occurs if the outputs are divided into two sets each with thirty-two elements. The first set includes the outcomes when four players choose the first strategy and two choose the second strategy, when all players choose the first strategy,and when all choose the second strategy. The rest of the outputs are in the second set. We could not find any game-theoretic situation with such a payoff structure. Thus, the results obtained so far are valid for up to six-player symmetric games which represent a game-theoretic situation and hence are the subject of game theory.

3.3. Reproducibility criterion as a benchmark

It is only when reproducibility criterion is satisfied, we can compare the outcomes of classical and quantum versions to draw conclusions on whether one has advantage over the other. The first thing the physical scheme should provide is unitary operators corresponding to classical pure strategies for a given entangled state. If there exists such operators then one can compare the outcomes for the pure strategies. Let us consider the entangled state $|W_N\rangle$ for which one cannot find $\{\hat{u}_k^1, \hat{u}_k^2\}$ satisfying the criterion. When a game is played using $|W_N\rangle$ with unitary operators chosen from the SU(2), the outcomes of the classical game in pure strategies cannot be obtained, because in the quantum pure strategy, the payoffs become a probability distribution over the entries of the classical payoff matrix. Therefore, comparing the quantum version using $|W_N\rangle$ with the classical game in pure strategies is not fair. In the same way, comparing quantum versions played with GHZ and $|W_N\rangle$ states is not fair either because for GHZ the payoffs delivered to the players are unique entries from the classical payoff table, contrary to those for $|W_N\rangle$. Thus, we think the reproducibility criterion constitutes a benchmark not only for the evaluation of entangled states and operators in quantum games but also for the comparison of classical games and their quantum versions.

4. Conclusion

In this paper, for the first time, we give the necessary and sufficient condition to play quantized version of classical games in a physical scheme. This condition is introduced here as the *reproducibility criterion* and it provides a fair basis to compare quantum versions of games with their classical counterparts. This benchmark requires the reproducibility of the results of the classical games in their quantum version. The SCR and WCR shows that a large class of multipartite entangled states cannot be used

in the quantum version of classical games; and the operators that might be used should have a special diagonalized form. Given two unitary operators $\{\hat{u}_k^1, \hat{u}_k^2\}$ corresponding to classical pure strategies and satisfying SCR and/or WCR, one can reproduce the results of classical games in pure strategies in the physical scheme. Moreover, provided that the players choose unitary operators in the space spanned by $\hat{u}_k^1$ and $\hat{u}_k^2$, mixed strategy results of classical games can be reproduced, too. The results are valid for a large class of entangled states, which can be prepared experimentally with the current level of technology, and multi-player extensions of interesting 2×2 games as well as for a large class of originally multiparty games.

Results also suggest that entangled states that cannot be used in two-strategy multi-player games due to SCR are good candidates for quantum information tasks (i.e, multi-party binary decision problems, etc) where anonymity of participants is required. SCR can be rephrased as the construction of complete orthogonal bases from an initially entangled state by local unitary operations when the parties are restricted to a limited number of operators. While this construction is possible for the states satisfying SCR, it is not possible for the others.

Extending this work to any generic game and the whole family of N-partite entangled states requires lengthy calculations and detailed classification of payoff structures which is beyond the scope of this paper. However, the results presented here are enough to show the importance of reproducibility criterion and the restrictions imposed by it.

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